On asymmetric generalization of the Weibull distribution by scale-location mixing of normal laws*

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Abstract: Two approaches are suggested to the definition of asymmetric generalized Weibull distribution. These approaches are based on the representation of the two-sided Weibull distributions as variance-mean normal mixtures or more general scale-location mixtures of the normal laws. Since both of these mixtures can be limit laws in limit theorems for random sums of independent random variables, these approaches can provide additional arguments in favor of asymmetric two-sided Webull-type models of statistical regularities observed in some problems related to stopped random walks, in particular, in problems of modeling the evolution of financial markets.

Key words: Webull distribution; scale-location mixture of normal laws; variance-mean normal mixture; random sum; stable distribution; exponential distribution

1 Introduction. The Weibull distribution

In probability theory and mathematical statistics it is conventional to use the term Weibull distribution for a special absolutely continuous probability distribution concentrated on the nonnegative half-line with exponential-power type decrease of the tail. It is called so after the Swedish scientist Waloddi Weibull (1887–1979) who suggested in 1939 to use this distribution in the study of the strength of materials [46, 47] and thoroughly analyzed this distribution in 1951 [48] demonstrating good perspectives of the application of this distribution to the description of many observed statistical regularities.

Let $\gamma > 0$. The distribution of the random variable W_{γ} :

$$P(W_{\gamma} < x) = [1 - e^{-x^{\gamma}}] \mathbf{1}(x \ge 0), \quad x \in \mathbb{R},$$
(1)

is called the Weibull distribution with shape parameter γ (here and in what follows the symbol $\mathbf{1}(C)$ denotes the indicator function of a set C).

However, Weibull was not the first to introduce this distribution. This distribution was for the first time described in 1927 by Maurice Fréchet [16] within the context of the study of the asymptotic behavior of extreme order statistics. Sometimes distribution (1) is called the Rosin–Rammler distribution after Paul Rosin and Erich Rammler, German scientists who were the first to use this distribution as a model of statistical regularities in the coal particles sizes in 1933 [41]. However, this term also does not completely correspond to the historical truth. In the paper [45] Dietrich Stoyan directly writes that this distribution was found by Paul Rosin, Erich Rammler, Karl Sperling [41, 42] and John Godolphin Bennett [8] within the context of particle size. It is well known that the Weibull distribution family is closed with respect to the operation of taking minimum of independent random

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variables. As it was demonstrated in [16, 15], due to this property the family of Weibull distributions is one of possible limit laws for extreme order statistics. B. V. Gnedenko found necessary and sufficient conditions for the convergence of the distributions of extreme order statistics to the Weibull distribution under linear normalization [17]. Therefore, this distribution is sometimes called the Weibull–Gnedenko distribution [25].

The case of small values of the parameter $\gamma \in (0,1]$ is of special interest for financial and some other applications, since Weibull distributions with such parameters (sometimes called *stretched exponential distributions* [31, 34, 35]) occupy an intermediate position between distributions with exponentially decreasing tails (such as exponential and gamma-distributions) and heavy-tailed Zipf-Pareto-type distributions with power-type decrease of tails.

In the paper [45] cited above, D. Stoyan notes that the name exponential power distribution would better fit to distribution (1), however, the latter term has been occupied by another absolutely continuous distribution with a similar behavior of tails [11, 27, 21], which, unlike distribution (1), has the exponential power density

$$\ell_{\gamma}(x) = \frac{\gamma}{2\Gamma(\frac{1}{\gamma})} e^{-|x|^{\gamma}}, \quad x \in \mathbb{R},$$

with $\gamma > 0$, whereas distribution (1) has the exponential power cumulative distribution function.

In this paper, for definiteness, for distribution (1) we will use the traditional term Weibull distribution.

The Weibull distribution is a special generalized gamma-distribution. It is widely used in survival analysis [14], in life insurance as a model of the lifetime distribution, in risk insurance as a model of the claim size distribution [23], in economics and financial mathematics as a model of asset returns distribution [3, 36, 37] and income distribution [7, 10], in reliability theory as a model of the distribution of time between failures [1, 32], in industrial technology as a model of the distribution of technological stages or time intervals between technological changes [43], in coal industry for the description of statistical regularities of particle sizes [41], in radio engineering and radiolocation, in meteorology, hydrology and many other fields, see, e. g., [25, 1, 32, 30, 24].

In particular, in [36] it was discovered that this distribution provides the best fit among others to the observed statistical regularities of the index S&P500, if its positive and negative increments are considered separately thus leading to the concept of a two-sided Weibull distribution with both positive and negative tails decreasing as an exponential power function. Some authors suggested to use the Weibull distribution as the errors distribution in range data modelling [12] or the distribution of trading duration [13].

In [44] it was proposed to use a symmetric two-sided Weibull distribution as an unconditional return distribution. A symmetric two-sided Weibull distribution was also mentioned in [33], but its properties were not explored. It should be noted that in these papers the attempts to introduce the two-sided Weibull distribution were rather formal and descriptive. In these papers as well as in [38, 39] the elementary properties of these models were described.

In the present paper, two new approaches are proposed to the definition of general asymmetric two-sided Weibull distribution by representing them as variance-mean and more general scale-location mixtures of normal laws.

Among all scale-location mixtures of normal laws, normal variance-mean mixtures occupy a special position. A distribution function F(x) is called a *normal variance-mean mixture*, if it has the form

$$F(x) = \int_0^\infty \Phi\left(\frac{x - \beta - \alpha z}{\sigma\sqrt{z}}\right) dG(z), \quad x \in \mathbb{R}.$$

with some $\beta \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\sigma \in (0, \infty)$, where $\Phi(x)$ is the standard normal distribution function, G(z) is a distribution function such that G(0) = 0. The class of normal variance-mean mixtures is very wide and contains, say, generalized hyperbolic laws [4, 5, 6] and generalized variance gamma distributions [49] which proved to provide excellent fit to statistical data in various fields from atmospheric turbulence to financial markets.

In normal variance-mean mixtures, mixing is performed with respect to both location and scale parameters. But since these parameters are tightly linked so that the location parameters (means) are proportional to the variances of the mixed laws, actually this is a one-parameter mixture.

Normal mixtures are limit distributions for sums of a random number of random variables. Therefore, these approaches can provide additional grounds for the adequacy of asymmetric two-sided Weibull distributions in practical problems modeled by stopped random walks, in particular, related to the description of the evolution of financial indexes.

The paper is organized as follows. Section 2 contains some auxiliary results dealing with product representations for Weibull-distributed random variables by normally and exponentially distributed random variables. In section 3 similar results are obtained for symmetric two-sided Weibull distributions. In section 4 the representation is obtained for the formal asymmetric two-sided Weibull distribution as a scale-location mixture of normal laws. Finally, in section 5 a new generalization of the Weibull distribution by variance-mean mixing of normal laws is proposed. Here a limit theorem for random sums of independent random variables is also proved illustrating that such Weibull-type distribution can be reasonable asymptotic approximation.

2 Product representations for Weibull-distributed random variables by normally and exponentially distributed random variables

Here we will show that the Weibull distribution admits several representations in the form of scale mixtures of some well-known probability distributions. For this purpose here some product representations for Weibull-distributed random variables by normally and exponentially distributed random variables will be proved. These representations will be used in the next section to prove that for $\gamma \in (0,1]$ the Weibull distribution is a mixed half-normal distribution and hence, it can be the limiting law for maximum random sums with finite variances.

Most results presented below actually concern special mixture representations for probability distributions. However, without any loss of generality, for the sake of visuality and compactness of formulations and proofs we will formulate the results in terms of the corresponding random variables assuming that all the random variables mentioned in what follows are defined on the same probability space $(\Omega, \mathfrak{A}, P)$.

It is obvious that W_1 is the random variable with the standard exponential distribution: $\mathsf{P}(W_1 < x) = \left[1 - e^{-x}\right] \mathbf{1}(x \ge 0)$. The Weibull distribution with $\gamma = 2$, that is, $\mathsf{P}(W_2 < x) = \left[1 - e^{-x^2}\right] \mathbf{1}(x \ge 0)$ is called the Rayleigh distribution [40].

The random variable with the standard normal distribution function $\Phi(x)$ will be denoted X,

$$P(X < x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz, \quad x \in \mathbb{R}.$$

Let $\Psi(x)$, $x \in \mathbb{R}$, be the distribution function of the maximum of the standard Wiener process on the unit interval, $\Psi(x) = 2\Phi(\max\{0,x\}) - 1$, $x \in \mathbb{R}$. It is easy to see that $\Psi(x) = P(|X| < x)$. Therefore, sometimes $\Psi(x)$ is said to determine the *half-normal* distribution.

The symbol $\stackrel{d}{=}$ denotes the coincidence of distributions.

Lemma 1. The relation

$$W_1 \stackrel{d}{=} \sqrt{2W_1}|X| \tag{2}$$

holds, where the random variables on the right-hand side are independent.

PROOF. For x > 0 we have

$$\begin{split} \mathsf{P} \big(|X| \sqrt{W_1} < x \big) &= \mathsf{E} \Psi \big(x / \sqrt{W_1} \big) = 2 \mathsf{E} \Phi \big(x / \sqrt{W_1} \big) - 1 = 2 \int_0^\infty \! \Phi \big(x / \sqrt{z} \big) d[1 - e^{-z}] - 1 = \\ &= 2 \int_0^\infty \! \left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}} \! \int_0^{x / \sqrt{z}} \! e^{-u^2/2} du \right] e^{-z} dz - 1 = \frac{\sqrt{2}}{\sqrt{\pi}} \! \int_0^\infty \! \int_0^{x / \sqrt{z}} \! e^{-u^2/2 - z} du dz = \end{split}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \int_0^{x^2/u^2} e^{-z} dz e^{-u^2/2} du = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \left(1 - \exp\left\{ -\frac{x^2}{u^2} \right\} \right) e^{-u^2/2} du =$$

$$= 1 - \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \exp\left\{ -\frac{u^2}{2} - \frac{x^2}{u^2} \right\} du = 1 - e^{-\sqrt{2}x} = \mathsf{P}(W_1 < \sqrt{2}x),$$

see, e. g., [20], formula 3.325. This is nothing else that the exponential distribution with parameter $\sqrt{2}$. The lemma is proved.

It is easy to see that if $\gamma > 0$ and $\gamma' > 0$, then $\mathsf{P}(W_{\gamma'}^{1/\gamma} \geqslant x) = \mathsf{P}(W_{\gamma'} \geqslant x^{\gamma}) = e^{-x^{\gamma\gamma'}} = \mathsf{P}(W_{\gamma\gamma'} \geqslant x)$, $x \geqslant 0$, that is, for any $\gamma > 0$ and $\gamma' > 0$

$$W_{\gamma\gamma'} \stackrel{d}{=} W_{\gamma'}^{1/\gamma}. \tag{3}$$

In particular, for any $\gamma > 0$ we have

$$W_{\gamma} \stackrel{d}{=} W_1^{1/\gamma}. \tag{4}$$

3 Product representations for Weibull-distributed random variables by random variables with stable distributions

The distribution function and density of the strictly stable distribution with the characteristic exponent α and parameter θ , corresponding to the characteristic function

$$\mathfrak{f}_{\alpha,\theta}(t) = \exp\left\{-|t|^{\alpha} \exp\left\{-\frac{1}{2}i\pi\theta\alpha \operatorname{sign}t\right\}\right\}, \quad t \in \mathbb{R},\tag{5}$$

with $0 < \alpha \le 2$ and $|\theta| \le \min\{1, \frac{2}{\alpha} - 1\}$ will be respectively denoted $G_{\alpha,\theta}(x)$ and $g_{\alpha,\theta}(x)$, $x \in \mathbb{R}$ (see, e. g., [50]).

From (5) it follows that the characteristic function of a symmetric ($\theta = 0$) strictly stable distribution has the form

$$\mathfrak{f}_{\alpha,0}(t) = e^{-|t|^{\alpha}}, \quad t \in \mathbb{R}. \tag{6}$$

LEMMA 2. Any symmetric strictly stable distribution with the characteristic exponent α is a scale mixture of normal laws with the one-sided strictly stable law $(\theta = 1)$ with the characteristic exponent $\alpha/2$ as the mixing distribution:

$$G_{\alpha,0}(x) = \int_0^\infty \Phi(x/\sqrt{z}) dG_{\alpha/2,1}(z), \quad x \in \mathbb{R}.$$
 (7)

PROOF. See, e. g., theorem 3.3.1 in [50].

In order to prove that any Weibull distribution with $\gamma \in (0,1]$ is a scale mixture of half-normal laws, we first make sure that any Weibull distribution with $\gamma \in (0,2]$ is a scale mixture of Rayleigh distributions.

Everywhere in what follows for $\gamma \in (0,1)$ we will use the notation $V_{\gamma} = 2S_{\gamma,1}^{-1}$, where $S_{\gamma,1}$ is a random variable with one-sided strictly stable density $g_{\gamma,1}(x)$.

Lemma 3. For any $\gamma \in (0,2]$ the product representation

$$W_{\gamma} \stackrel{d}{=} W_2 \sqrt{V_{\gamma/2}} \tag{8}$$

holds, where the random variables on the right-hand side of (8) are independent.

PROOF. Write relation (7) in terms of characteristic functions with the account of (6):

$$e^{-|t|^{\alpha}} = \int_0^{\infty} \exp\{-\frac{1}{2}t^2z\}g_{\alpha/2,1}(z)dz, \quad t \in \mathbb{R}.$$
 (9)

Formally setting in (9) |t| = x with $x \ge 0$ being an arbitrary nonnegative number, we obtain

$$P(W_{\gamma} > x) = e^{-x^{\gamma}} = \int_{0}^{\infty} \exp\{-\frac{1}{2}x^{2}z\}g_{\gamma/2,1}(z)dz. \tag{10}$$

At the same time, we obviously have

$$P(W_2\sqrt{V_{\gamma/2}} > x) = P(W_2 > x\sqrt{\frac{1}{2}S_{\gamma/2,1}}) = \int_0^\infty \exp\{-\frac{1}{2}x^2z\}g_{\gamma/2,1}(z)dz. \tag{11}$$

Since the right-hand sides of (10) and (11) identically (in $x \ge 0$) coincide, the left-hand sides of these relations identically coincide as well. The lemma is proved.

LEMMA 4. For any $\gamma \in (0,1]$ the Weibull distribution with parameter γ is a mixed exponential distribution:

$$\mathsf{P}(W_{\gamma} > x) = \int_0^\infty e^{-\frac{1}{2}zx} g_{\gamma,1}(z) dz, \quad x \geqslant 0.$$

PROOF. From (3) it follows that $W_2 \stackrel{d}{=} \sqrt{W_1}$. Therefore lemma 3 implies that for $\gamma \in (0,1]$

$$W_{2\gamma} \stackrel{d}{=} W_2 \sqrt{V_{\gamma}} \stackrel{d}{=} \sqrt{W_1 V_{\gamma}}$$

or, with the account of (3),

$$W_{\gamma} \stackrel{d}{=} W_{2\gamma}^2 \stackrel{d}{=} W_1 V_{\gamma},\tag{12}$$

that is.

$$e^{-x^{\gamma}} = \mathsf{P}(W_{\gamma} > x) = \mathsf{P}(W_1 > \frac{1}{2}S_{\gamma,1}x) = \int_0^{\infty} e^{-\frac{1}{2}zx} g_{\gamma,1}(z)dz, \quad x \geqslant 0.$$

The lemma is proved.

COROLLARY 1. Any Weibull distribution with $\gamma \in (0,1]$ is infinitely divisible.

PROOF. This statement is actually a particular case of a result on generalized gamma-convolutions in [9] (see corollary 2 there). However, we can also see that the desired assertion immediately follows from (12) and the result of Goldie [19] stating that the product of two independent non-negative random variables is infinitely divisible if one of the two is exponentially distributed.

Theorem 1. For any $\gamma \in (0,1]$ the Weibull distribution with parameter γ is a scale mixture of half-normal laws:

$$W_{\gamma} \stackrel{d}{=} |X| \sqrt{2W_1 V_{\gamma}^2},\tag{13}$$

where the random variables on the right-hand side of (13) are independent.

PROOF. To prove the desired result it suffices to use the representation for W_1 proved in lemma 1 on the right-hand side of (12) and obtain (13). The theorem is proved.

Theorem 1 makes it possible to represent the Weibull distribution with $\gamma \in (0,1]$ as

$$\mathsf{P}(W_{\gamma} < x) = \mathsf{E}\Psi \big(x/\sqrt{2W_1V_{\gamma}^2} \big) = \int_0^{\infty} \! \Psi \big(x/\sqrt{y} \big) dH_{\gamma}(y), \quad x \in \mathbb{R},$$

where

$$H_{\gamma}(y) = \mathsf{P}(2W_1V_{\gamma}^2 < y) = \mathsf{P}(W_1 < \frac{1}{8}yS_{\gamma,1}^2) = 1 - \int_0^\infty \exp\left\{-\frac{1}{8}yz^2\right\} dG_{\gamma,1}(z), \quad y \geqslant 0. \tag{14}$$

The following result generalizes lemma 4 and establishes that the Weibull distribution with an arbitrary positive shape parameter γ is a scale mixture of the Weibull distribution with an arbitrary positive shape parameter $\delta > \gamma$.

Theorem 2. Let $\delta > \gamma > 0$. Then

$$W_{\gamma} \stackrel{d}{=} W_{\delta} \cdot V_{\alpha}^{1/\delta}$$

where $\alpha = \gamma/\delta \in (0,1)$ and the random variables on the right-hand side are independent.

PROOF. For any $\delta > \gamma > 0$, denoting $\alpha = \gamma/\delta$ (as this is so, we obviously have $\alpha \in (0,1)$), for any $x \in \mathbb{R}$, from lemma 4 we obtain

$$\begin{split} \mathsf{P}(W_\gamma>x) &= e^{-x^\gamma} = e^{-x^{\delta\alpha}} = \mathsf{P}(W_\alpha>x^\delta) = \mathsf{P}(W_1>\tfrac{1}{2}S_{\alpha,1}x^\delta) = \\ &= \int_0^\infty e^{-\tfrac{1}{2}zx^\delta} g_{\alpha,1}(z) dz = \int_0^\infty \mathsf{P}\big(W_\delta>x(\tfrac{1}{2}z)^{1/\delta}\big) g_{\alpha,1}(z) dz = \mathsf{P}(W_\delta\cdot V_\alpha^{1/\delta}>x), \end{split}$$
 Q. E. D.

REMARK 1. If $0 < \gamma < \delta \le 2$, then the result of theorem 2 directly follows from theorem 2.3.1 in [50] by virtue of the formal coincidence of the characteristic function of a strictly stable law with the complementary distribution function of the Weibull law with the corresponding parameter (see the proof of lemma 3).

The representation of the Weibull distribution with $\gamma \in (0,1)$ in the form of a mixed exponential distribution (see lemma 4) yields the following by-product result concerning the explicit representation of the moments of symmetric or one-sided strictly stable distributions.

COROLLARY 2. Let $S_{\gamma,1}$ be a positive random variable with the strictly stable distribution with the characteristic exponent $\gamma \in (0,1)$ and $\theta = 1$. For $\beta \in (0,\gamma)$ we have

$$\mathsf{E} S_{\gamma,1}^{\beta} = \frac{2^{\beta} \Gamma(1 - \beta/\gamma)}{\Gamma(1 - \beta)}.$$

PROOF. For any $\delta > -\gamma$ we have $\mathsf{E}W_{\gamma}^{\delta} = \Gamma(1+\delta/\gamma)$. For such δ from lemma 3 it follows that $\mathsf{E}W_{\gamma}^{\delta} = 2^{\delta} \mathsf{E}W_{1}^{\delta} \mathsf{E}S_{\gamma,1}^{-\delta}$. Setting $\delta = -\beta > -\gamma$ we obtain the desired assertion.

The statement of lemma 2 can be rewritten as $S_{\alpha,0} \stackrel{d}{=} X \sqrt{S_{\alpha/2,1}}$ with the random variables on the right-hand side being independent. Therefore, corollary 2, in turn, implies

COROLLARY 3. Let $S_{\alpha,0}$ be a random variable with the symmetric strictly stable distribution with the characteristic exponent $\alpha \in (0,2)$ (see (6)). Then for $\beta < \alpha < 2$ we have

$$\mathsf{E}|S_{\alpha,0}|^{\beta} = \frac{2^{\beta}}{\sqrt{\pi}} \cdot \frac{\Gamma((\beta+1)/2)\Gamma((\alpha-\beta)/\alpha)}{\Gamma((2-\beta)/\beta)}.$$

4 Product representations for random variables with symmetric twosided Weibull distributions

Let $\gamma > 0$. The distribution of the random variable \widetilde{W}_{γ} :

$$P(\widetilde{W}_{\gamma} < x) = \frac{1}{2}e^{-|x|^{\gamma}}\mathbf{1}(x < 0) + \left[1 - \frac{1}{2}e^{-x^{\gamma}}\right]\mathbf{1}(x \ge 0)$$
(15)

is called the *symmetric two-sided Weibull distribution* with shape parameter γ . Distribution (15) was introduced in the paper [44] as a heavy-tailed model for the evaluation of financial risks. Further generalizations and references can be found, for example, in the recent works [38, 39].

It is easy to see that if W_{γ} is a random variable with the usual (one-sided) Weibull distribution (1) and U is a random variable taking values -1 and +1 with probabilities $\frac{1}{2}$ each and independent of W_{γ} , then $\widetilde{W}_{\gamma} \stackrel{d}{=} UW_{\gamma}$ and hence, $|\widetilde{W}_{\gamma}| \stackrel{d}{=} W_{\gamma}$.

Moreover, from theorem 1 it obviously follows that for $\gamma \in (0,1]$ we have

$$\widetilde{W}_{\gamma} \stackrel{d}{=} X \sqrt{2W_1V_{\gamma}^2}$$

where the random variables on the right-hand side are independent.

COROLLARY 4. For any $\gamma \in (0,1]$ the symmetric two-sided Weibull distribution with parameter γ is a scale mixture of normal laws:

$$\mathsf{P}(\widetilde{W}_{\gamma} < x) = \mathsf{E}\Phi\big(x/\sqrt{2W_1V_{\gamma}^2}\big) = \int_0^{\infty} \Phi\big(x/\sqrt{y}\big) dH_{\gamma}(y), \quad x \in \mathbb{R},$$

where, as in (14),

$$H_{\gamma}(y) = 1 - \int_{0}^{\infty} \exp\left\{-\frac{1}{8}yz^{2}\right\} dG_{\gamma,1}(z), \quad y \geqslant 0.$$

Note that here the mixing distribution $H_{\gamma}(x)$ is exactly the same as in theorem 1, furthermore, this distribution is absolutely continuous. It is easy to see that the corresponding density has the form

$$h_{\gamma}(x) = \frac{1}{8} \int_{0}^{\infty} z^{2} \exp\left\{-\frac{1}{8}xz^{2}\right\} g_{\gamma,1}(z) dz, \quad x > 0.$$

Despite the fact that, in general, the mixing distribution $H_{\gamma}(x)$ cannot be expressed via elementary functions, it is possible to trace the asymptotic behavior of its tail. Namely, in [2] the following result was proved connecting the tail behavior of a normal scale mixture with that of the corresponding mixing distribution.

Lemma 5 [2]. Assume that a distribution function F(x) is a scale mixture of zero-mean normal laws,

$$F(x) = \int_0^\infty \Phi\left(\frac{x}{\sqrt{u}}\right) dQ(y), \quad x \in \mathbb{R},$$

with Q(0) = 0. Let L(x) be a positive slowly varying function, that is, $L : \mathbb{R} \to \mathbb{R}_+$ so that for any p > 0

$$\lim_{x \to \infty} \frac{L(px)}{L(x)} = 1.$$

Let $\rho \in (0,2)$, $\beta > 0$. Then

$$\liminf_{x\to\infty}\frac{\ln[1-F(x)]}{x^\rho L(x)}=-\frac{1}{\beta}$$

if and only if

$$\liminf_{u \to \infty} \frac{\ln[1 - Q(u)]}{u^{\frac{\rho}{2-\rho}} [L(u^{\frac{\rho}{2-\rho}})]^{\frac{2}{2-\rho}}} = -\frac{1}{2\beta^{\frac{2}{2-\rho}}}.$$

Let $\gamma \in (0,1]$. We obviously have $\mathsf{P}(\widetilde{W}_{\gamma} > x) = \frac{1}{2}e^{-x^{\gamma}}$. Therefore from corollary 4 and lemma 5 with $\rho = \gamma$, $\beta = 1$ and $L(x) \equiv 1$ it follows that

$$1 - H_{\gamma}(x) \sim \exp\left\{-\frac{1}{2}x^{\frac{\gamma}{2-\gamma}}\right\}, \quad x \to \infty.$$

that is, the tail of the mixing distribution also decreases as an exponential power function.

Obviously, \widetilde{W}_1 is the random variable with the Laplace distribution

$$L(x) \equiv P(\widetilde{W}_1 < x) = \frac{1}{2}e^x \mathbf{1}(x < 0) + \left[1 - \frac{1}{2}e^{-x}\right] \mathbf{1}(x \ge 0).$$

It can be made sure that the product representation

$$\widetilde{W}_1 \stackrel{d}{=} X \sqrt{2W_1}$$

holds, where the random variables on the right-hand side are independent (see, e. g., [26], p. 578-579). Then corollary 4, in turn, implies

COROLLARY 5. For any $\gamma \in (0,1]$ the symmetric two-sided Weibull distribution with parameter γ is a scale mixture of Laplace distributions:

$$\mathsf{P}(\widetilde{W}_{\gamma} < x) = \mathsf{E}L\left(\frac{1}{2}xS_{\gamma,1}\right) = \int_{0}^{\infty} L\left(\frac{1}{2}xy\right)g_{\gamma,1}(y)dy, \quad x \in \mathbb{R}.$$

Actually corollary 5 is a particular case of the following more general statement which is an analog of theorem 2 for symmetric two-sided Weibull distributions.

Theorem 3. Let $\delta > \gamma > 0$. Then

$$\widetilde{W}_{\gamma} \stackrel{d}{=} \widetilde{W}_{\delta} \cdot V_{\alpha}^{1/\delta},$$

where $\alpha = \gamma/\delta \in (0,1)$, and the random variables on the right-hand side are independent.

PROOF. As it has been noted above, $|\widetilde{W}_{\gamma}| \stackrel{d}{=} W_{\gamma}$ for any $\gamma > 0$. Then by theorem 2

$$|\widetilde{W}_{\gamma}| \stackrel{d}{=} W_{\gamma} \stackrel{d}{=} W_{\delta} \cdot V_{\alpha}^{1/\delta} \stackrel{d}{=} |\widetilde{W}_{\delta}| \cdot V_{\alpha}^{1/\delta} = |\widetilde{W}_{\delta} \cdot V_{\alpha}^{1/\delta}|.$$

Hence, since the distributions of \widetilde{W}_{γ} and \widetilde{W}_{δ} are symmetric, for arbitrary x>0 we have

$$\mathsf{P}(\widetilde{W}_{\gamma} < x) = \tfrac{1}{2} \big[\mathsf{P}(|\widetilde{W}_{\gamma}| < x) + 1 \big] = \tfrac{1}{2} \big[\mathsf{P}(|\widetilde{W}_{\delta} \cdot V_{\alpha}^{1/\delta}| < x) + 1 \big] = \mathsf{P}(\widetilde{W}_{\delta} \cdot V_{\alpha}^{1/\delta} < x).$$

And if x < 0, then -x > 0 so that according to what has already been proved,

$$\mathsf{P}(\widetilde{W}_{\gamma} < x) = 1 - \mathsf{P}(\widetilde{W}_{\gamma} < -x) = 1 - \mathsf{P}(\widetilde{W}_{\delta} \cdot V_{\alpha}^{1/\delta} < -x) = \mathsf{P}(\widetilde{W}_{\delta} \cdot V_{\alpha}^{1/\delta} < x).$$

The theorem is proved.

5 Formal asymmetric two-sided Weibull distribution as a scalelocation mixture of normal laws

We begin this section with the following formal definition. Let a_1 and a_2 be two finite positive numbers, $\gamma > 0$.

DEFINITION 1. A random variable $W_{a_1,a_2;\gamma}$ will be said to have the asymmetric Weibull distribution of the first kind $\mathfrak{W}_I(x)$ with parameters a_1, a_2, γ , if its distribution function has the form

$$\mathfrak{W}_{I}(x) = \mathsf{P}(W_{a_{1}, a_{2}; \gamma} < x) = \begin{cases} \frac{a_{1}}{a_{1} + a_{2}} \cdot e^{-(a_{2}|x|)^{\gamma}}, & x \leq 0\\ 1 - \frac{a_{2}}{a_{1} + a_{2}} \cdot e^{-(a_{1}x)^{\gamma}}, & x > 0. \end{cases}$$
(16)

Each of positive and negative branches of the distribution $\mathfrak{W}_I(x)$ formally coincides with the classical Weibull distribution. Our aim in this section is to obtain normal mixture representations for distribution (16). We will construct this representation in two stages. On the first stage we will construct an asymmetric normal-mixture-type representation for the Laplace (two-sided exponential) distribution.

Let a_1 and a_2 be two finite positive numbers. We will say that a random variable Λ_{a_1,a_2} has the asymmetric Laplace distribution with parameters a_1 and a_2 , if $\Lambda_{a_1,a_2} \stackrel{d}{=} W_{a_1,a_2;1}$, where the distribution of $W_{a_1,a_2;1}$ is defined by relation (16) with $\gamma = 1$.

It is easy to see that the probability density $\ell_{a_1,a_2}(x)$ corresponding to the distribution function $L_{a_1,a_2}(x) = P(\Lambda_{a_1,a_2} < x)$ has the form

$$\ell_{a_1,a_2}(x) = \begin{cases} \frac{a_1 a_2}{a_1 + a_2} \cdot e^{a_2 x} & \text{for } x \leq 0\\ \frac{a_1 a_2}{a_1 + a_2} \cdot e^{-a_1 x} & \text{for } x > 0. \end{cases}$$

The asymmetric Laplace distribution is a popular model widely used in many fields, see, e. g., [29]. Show that this distribution is a special variance-mean normal mixture.

LEMMA 6. Let $\mu \in \mathbb{R}$, $\sigma^2 \in (0, \infty)$, $\lambda \in (0, \infty)$. Assume that the random variable Y is representable in the form

$$Y \stackrel{d}{=} \frac{\sigma}{\sqrt{\lambda}} \cdot X \sqrt{W_1} + \mu \cdot \frac{W_1}{\lambda},$$

where the random variable X has the standard normal distribution and the random variable W_1 has the standard exponential distribution (so that W_1/λ has the exponential distribution with parameter λ), and the random variables X and W_1 are independent. Then $Y \stackrel{d}{=} \Lambda_{a_1,a_2}$, that is,

$$\mathsf{P}(Y < x) = \mathsf{E}\Phi\left(\frac{\lambda x - \mu W_1}{\sigma\sqrt{\lambda W_1}}\right) = L_{a_1, a_2}(x), \quad x \in \mathbb{R},$$

where

$$a_1 = \frac{1}{\sqrt{\mu^2 + 2\lambda\sigma^2 + \mu}}, \quad a_2 = \frac{1}{\sqrt{\mu^2 + 2\lambda\sigma^2 - \mu}}.$$

PROOF. It is easy to see that, by the independence of X and W_1 , the characteristic function of Y has the form

$$\mathsf{E} e^{itY} = \mathsf{E} e^{it(\sigma X\sqrt{U} + \mu U)} = \lambda \int_0^\infty \exp\big\{z\big(it\mu - \frac{1}{2}\sigma^2t^2 - \lambda\big)\big\} dz = \frac{\lambda}{it\mu - \frac{1}{2}\sigma^2t^2 - \lambda}, \quad t \in \mathbb{R}. \tag{17}$$

It remains to make sure that characteristic function (17) corresponds to the asymmetric Laplace distribution. To obtain a simpler form of the right-hand side of (17), change the parameters:

$$\begin{cases} w - v = \frac{\mu}{\lambda}, \\ v \cdot w = \frac{\sigma^2}{2\lambda}. \end{cases}$$
 (18)

From the first equation (18) we obtain

$$w = v + \frac{\mu}{\lambda}$$

whereas the second equation yields

$$v\left(v + \frac{\mu}{\lambda}\right) = \frac{\sigma^2}{2\lambda}$$

or

$$v^2 + \frac{\mu}{\lambda}v - \frac{\sigma^2}{2\lambda} = 0.$$

System (18) has two solutions with respect to v:

$$v_1 = -\frac{\mu}{2\lambda} + \frac{1}{2}\sqrt{\frac{\mu^2}{\lambda^2} + \frac{2\sigma^2}{\lambda}}, \quad v_2 = -\frac{\mu}{2\lambda} - \frac{1}{2}\sqrt{\frac{\mu^2}{\lambda^2} + \frac{2\sigma^2}{\lambda}}.$$

One of them, v_1 , is positive. As this is so, $w_1 = v_1 + \frac{\mu}{\lambda}$ is also positive. The parameters $v = v_1$ and $w = w_1$ will be used in what follows.

With the new parametrization, characteristic function (17) takes the form

$$\mathsf{E}e^{itY} = \frac{1}{(1-iwt)} \cdot \frac{1}{(1+ivt)}.$$

Notice that $\frac{1}{1-iwt}$ is the characteristic function of the exponential distribution with parameter $a_1 = \frac{1}{w}$ corresponding to the density

$$p_1(x) = \begin{cases} 0, & x \le 0 \\ a_1 e^{-a_1 x}, & x > 0. \end{cases}$$

At the same time $\frac{1}{1+ivt}$ is the characteristic function corresponding to the density

$$p_2(x) = \begin{cases} a_2 e^{a_2 x}, & x \le 0 \\ 0, & x > 0, \end{cases}$$

where $a_2 = \frac{1}{v}$. Hence, g(t) is the characteristic function of the convolution p(x) of the densities $p_1(x)$ up $p_2(x)$ which has the following form: for $x \leq 0$

$$p(x) = \int_{-\infty}^{x} a_1 e^{-a_1(x-y)} a_2 e^{a_2 y} dy = \frac{a_1 a_2}{a_1 + a_2} e^{a_2 x},$$

and for x > 0

$$p(x) = \int_{-\infty}^{0} a_1 e^{-a_1(x-y)} a_2 e^{a_2 y} dy = \frac{a_1 a_2}{a_1 + a_2} e^{-a_1 x},$$

that is, $p(x) = \ell_{a_1,a_2}(x)$, $x \in \mathbb{R}$. In other words, this density corresponds to asymmetric Laplace distribution (1). Returning to the original parameters we obtain

$$a_1 = (\sqrt{\mu^2 + 2\lambda\sigma^2} + \mu)^{-1}, \quad a_2 = (\sqrt{\mu^2 + 2\lambda\sigma^2} - \mu)^{-1}$$

The lemma is proved.

From lemma 4 by formal calculation it follows that for $\gamma \in (0,1]$ and $y \ge 0$

$$\mathsf{P}(\Lambda_{a_1,a_2} \cdot V_{\gamma} > y) = \frac{a_2}{a_1 + a_2} \int_0^\infty e^{-a_1 y z} g_{\gamma,1}(z) dz = \frac{a_2 e^{-(a_1 y)^{\gamma}}}{a_1 + a_2},\tag{19}$$

and for $\gamma \in (0,1]$ and y < 0

$$\mathsf{P}(\Lambda_{a_1,a_2} \cdot V_{\gamma} < y) = \frac{a_1}{a_1 + a_2} \int_0^\infty e^{-a_2|y|z} g_{\gamma,1}(z) dz = \frac{a_1 e^{-(a_2|y|)^{\gamma}}}{a_1 + a_2}. \tag{20}$$

Now from lemma 5 and relations (19) and (20) we obtain the following statement.

THEOREM 3. Let $\gamma \in (0,1]$, $\mu \in \mathbb{R}$, $\sigma^2 \in (0,\infty)$, $\lambda \in (0,\infty)$. Assume that a random variable Z is representable in the form

$$Z \stackrel{d}{=} \left(\frac{\sigma}{\sqrt{\lambda}} \cdot X \sqrt{W_1} + \frac{\mu W_1}{\lambda} \right) \cdot V_{\gamma},$$

where the random variable X has the standard normal distribution, the random variable W_1 has the standard exponential distribution (so that W_1/λ has the exponential distribution with parameter λ), $V_{\gamma} = 2S_{\gamma,1}^{-1}$, where $S_{\gamma,1}$ is the random variable with the one-sided strictly stable density $g_{\gamma,1}(x)$, moreover, X, W_1 u $S_{\gamma,1}$ are independent. Then $Z \stackrel{d}{=} W_{a_1,a_2;\gamma}$, that is,

$$P(Z < x) = E\Phi\left(\frac{\lambda x - \mu W_1 V_{\gamma}}{\sigma \sqrt{\lambda W_1} V_{\gamma}}\right) = P(W_{a_1, a_2; \gamma} < x) = \mathfrak{W}_I(x), \quad x \in \mathbb{R}, \tag{21}$$

where

$$a_1 = \frac{1}{\sqrt{\mu^2 + 2\lambda\sigma^2 + \mu}}, \quad a_2 = \frac{1}{\sqrt{\mu^2 + 2\lambda\sigma^2 - \mu}}.$$

6 Asymmetric generalization of the two-sided Weibull distribution by variance-mean mixing of normal laws

Although the asymmetric generalization $\mathfrak{W}_I(x)$ of the two-sided Weibull distribution introduced in the preceding section is a scale-location mixture of normal laws, it is not so easy to give an example of a simple limit scheme, say, for random sums of independent random variables with such a distribution as a limit law, since the random shift and scale parameters in (21) are linked in a non-trivial way. However, if we change the definition of the asymmetric Weibull distribution in a reasonable way, then such a limit scheme can be constructed rather easily. For this purpose we will use a representation of the Weibull-type distribution as a variance-mean normal mixture based on the representation of the symmetric two-sided Weibull law obtained in corollary 4.

DEFINITION 2. Let $\mu \in \mathbb{R}$, $\sigma > 0$, $\gamma \in (0,1]$. A random variable $W_{\mu,\sigma;\gamma}^*$ will be said to have the asymmetric Weibull distribution of the second kind $\mathfrak{W}_{II}(x)$ with parameters μ , σ and γ , if its distribution function has the form of the variance-mean normal mixture

$$\mathfrak{W}_{II}(x) = \int_0^\infty \Phi\left(\frac{x - \mu z}{\sigma\sqrt{z}}\right) dH_\gamma(z), \quad x \in \mathbb{R},$$

where the mixing distribution function $H_{\gamma}(z)$ is defined in (14).

From corollary 4 it obviously follows that with $\mu = 0$, the asymmetric two-sided Weibull distribution of the second kind $\mathfrak{W}_{II}(x)$ has the form (15). But if $\mu \neq 0$, then this distribution cannot be expressed in terms of elementary functions. Nevertheless, it is rather easy to formulate a limit theorem for random sums of independent identically distributed random variables with finite variances in which the asymmetric two-sided Weibull distribution of the second kind $\mathfrak{W}_{II}(x)$ turns out to be the limit law.

Let $\{X_{n,j}\}_{j\geqslant 1}$, $n=1,2,\ldots$, be double array of row-wise independent and identically distributed random variables. Let $\{N_n\}_{n\geqslant 1}$ be a sequence of nonnegative integer-valued random variables such that for each $n\geqslant 1$ the random variables $N_n,X_{n,1},X_{n,2},\ldots$ are independent. For any $n,k\in\mathbb{N}$ let

$$S_{n,k} = X_{n,1} + \ldots + X_{n,k}.$$

To avoid misunderstanding, assume that $\sum_{j=1}^{0} = 0$. The symbol \Longrightarrow will denote the convergence in distribution.

In [28] the following statement was proved.

LEMMA 7. Assume that there exist a sequence of natural numbers $\{k_n\}_{n\geqslant 1}$ and numbers $\mu\in\mathbb{R}$ and $\sigma>0$ such that

$$\mathsf{P}\big(S_{n,k_n} < x\big) \Longrightarrow \Phi\Big(\frac{x-\mu}{\sigma}\Big). \tag{22}$$

Assume that $N_n \to \infty$ in probability. Then the distributions of the random sums S_{N_n} converge to some distribution function F(x):

$$P(S_{n,N_n} < x) \Longrightarrow F(x),$$

if and only if there exists a distribution function Q(x) such that Q(0) = 0,

$$F(x) = \int_0^\infty \Phi\left(\frac{x - \mu z}{\sigma\sqrt{z}}\right) dQ(z),$$

and

$$P(N_n < xk_n) \Longrightarrow Q(x).$$

REMARK 2. Condition (22) holds in the following rather general situation. Assume that the random variables $X_{n,j}$ have finite variances. Also assume that for each n and j

$$X_{n,j} = X_{n,j}^* + \mu_n,$$

where $\mu_n \in \mathbb{R}$ and $X_{n,j}^*$ is a random variable with $\mathsf{E} X_{n,j}^* = 0$, $\mathsf{D} X_{n,j}^* = \sigma_n^2 < \infty$, so that $\mathsf{E} X_{n,1} = \mu_n$ $\mathsf{D} X_{n,1} = \sigma_n^2$. Assume that $\mu_n k_n \to \mu \in \mathbb{R}$ $\mathsf{M} k_n \sigma_n^2 \to \sigma^2 \in (0,\infty)$ as $n \to \infty$. Then according to the classical result on necessary and sufficient conditions for the convergence of the distributions of independent identically distributed random variables with finite variances to the normal law in the double array limit scheme (see, e. g., [18]), we can see that convergence (22) takes place if and only if the Lindeberg condition holds:

$$\lim_{n \to \infty} k_n \mathsf{E}(X_{n,1}^*)^2 \mathbf{1}(|X_{n,1}^*| \ge \varepsilon) = 0$$

for any $\varepsilon > 0$.

From lemma 7 and definition 2 we immediately obtain the following statement establishing necessary and sufficient conditions for the convergence of the distributions of random sums of independent identically distributed random variables to the asymmetric two-sided Weibull distribution of the second kind $\mathfrak{W}_{II}(x)$.

THEOREM 4. Assume that there exist a sequence of natural numbers $\{k_n\}_{n\geqslant 1}$ and numbers $\mu\in\mathbb{R}$ and $\sigma>0$ such that convergence (22) takes place. Assume that $N_n\to\infty$ in probability. Then the distributions of random sums S_{N_n} of independent identically distributed random variables converge to to the asymmetric two-sided Weibull distribution of the second kind $\mathfrak{W}_{II}(x)$ with parameters μ , σ , γ :

$$P(S_{n,N_n} < x) \Longrightarrow \mathfrak{W}_{II}(x),$$

if and only if

$$\mathsf{P}(N_n < xk_n) \Longrightarrow H_{\gamma}(x),$$

where the distribution function $H_{\gamma}(x)$ is defined in (14).

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